Q1. **(True or False)** Please circle the correct answer. Each question worths 0.5 points. *You do NOT need to explain your answer.*

(i) Suppose x, y, z are some vectors in an inner product space $(V, \langle\ ,\ \rangle)$ such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.

TRUE FALSE

(ii) Suppose y, z are some vectors in an inner product space $(V, \langle\ ,\ \rangle)$ such that $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

TRUE FALSE

(iii) Any orthogonal subset of vectors in an inner product space is linearly independent.

It may contain the zero vector. TRUE FALSE

(iv) If W is a subspace of a finite dimensional inner product space, then $W = (W^{\perp})^{\perp}$.

See Section 6.2 Q13(c). TRUE FALSE

(v) Suppose x, y are vectors in a complex inner product space. If $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then $S = \{x, y\}$ must be orthogonal.

Consider $S = \{x, ix\}$, where x is a unit vector. TRUE FALSE

(vi) Any finite dimensional inner product space possesses an orthonormal basis.

TRUE FALSE

Q2. Let W be the subspace given by the span of the linearly independent subset $S = \{u_1, u_2, u_3\} \subset \mathbb{R}^4$ where

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 \\ 3 \\ 9 \\ 0 \end{pmatrix}.$$

(i) (3 points) Apply Gram-Schmidt process to the subset S and normalize it to obtain an *orthonormal* basis for W.

Let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix} - \frac{4}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix} 2 \\ 3 \\ 9 \\ 0 \end{pmatrix} - \frac{8}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - \frac{24}{8}\begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

An orthonormal basis for W is $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

i.e. $\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\}.$

(ii) (2 point) Find the unique vector $z \in W$ which is closest to the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Denote the orthonormal basis in (i) by $\{v_1', v_2', v_3'\}$.

The unique vector in W which is closest to $e_1 = (1,0,0,0)^T$ is:

$$z = \langle e_1, v_1' \rangle v_1' + \langle e, v_2' \rangle v_2' + \langle e_1, v_3' \rangle v_3'.$$

$$= \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Q3. (2 points) Prove the *parallelogram law*: for any x, y in an inner product space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

$$\|x+y\|^2 + \|x-y\|^2$$

$$= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$+ \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + (-1)^2 \langle y, y \rangle$$

$$= 2\langle x, x \rangle + 2\langle y, y \rangle$$

$$= 2\|x\|^2 + 2\|y\|^2$$

—END OF QUIZ 2—